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CHESTER B. SENSENIG

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USING A NON-LINEAR THEORY

Chester B. Sensenig

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1. Introduction

The object of this paper is to study the stability of equilibrium of an isotropic circular cylinder when compressed along its curved lateral surface. The compression on the curved part of the boundary is assumed to be such that the generators of the cylinder remain vertical straight lines, but so that no shear stress is developed; it is as though the cylinder were compressed by shrinking a very stiff greased ring down on it. The plane faces are assumed to be free of stress.

The problem is treated by making use of a non-linear three-dimensional theory of elasticity, of the same sort as treated in the book of Green and Zerna. The formulation of the theory used in this report follows that of Fritz John [1] in using Lagrange coordinates and a matrix notation. The essential point is that no restrictive assumptions are made concerning the thickness of the plate: the problem is treated as three-dimensional throughout. The displacements and strains are not assumed to be small. The strain energy function, from which the stress-strain relations come, could in principle be chosen arbitrarily without causing any essential difficulties; within this

report a special, but for many materials reasonable, choice for the strain energy function is made.

When the cylinder is compressed under the conditions stated above, a very simple state of strain, i.e., uniform plane strain, satisfies all conditions of the exact non-linear theory. A small perturbation with respect to this simple state is then made and the question is raised as to whether other possible equilibrium states - buckled states, in other words - exist within the framework of the basic three-dimensional theory, and what pressure is needed to cause them. Such critical pressures are found, in the present case (infinitely many of them in fact), together with the corresponding modes of buckling. This report is restricted, however, to buckled modes with symmetry about the axis of the cylinder. For these modes the critical buckling pressures and modes are given. They are the same as those furnished by the classical thin plate theory in the limit case of small thickness of the cylinder.

It should be pointed out specifically that these results are different from those given in an earlier report by S. Lubkin [2]. Lubkin treats the buckling of a hollow cylinder (as well as the hollow sphere and the rectangular solid), but he considers buckling modes in the plane of the faces of the cylinder, i.e., cases in which all quantities are independent of the coordinate along the axis of the cylinder.

2. Brief Description of the Non-Linear Theory

Let $x' = (x'_1, x'_2, x'_3)$ and $x = (x_1, x_2, x_3)$ be the position vectors to the same material particle in the strained and unstrained states respectively. The Lagrange coordinates, $x_i (i = 1, 2, 3)$, are used throughout this paper as independent variables, and the Eulerian

coordinates x_i^j ($i = 1, 2, 3$) are always used as dependent variables.

Let $p_{ij} = \partial x_i^j / \partial x_j$ ($i, j = 1, 2, 3$), and let p be the matrix (p_{ij}) . In the theory which we are using the material is described by a strain energy function u which is a function of p .

Let $q_{ij} = \partial u / \partial p_{ij}$ ($i, j = 1, 2, 3$), and let q be the matrix (q_{ij}) . If we neglect body forces, the equilibrium equations in Lagrange form are $q_{ij,j} = 0$ ($i = 1, 2, 3$) where $()_{,j} = \partial () / \partial x_j$, and we will use the usual summation convention throughout.

The stress matrix $\mathcal{C}$ is given by $\mathcal{C} = 1/|p| \, qp^*$ where $|p|$ is the determinant of p and $*$ denotes the transpose.

Let S and S' denote the same material surface in the unstrained and strained states respectively. Let ξ and ξ' be unit outer normal vectors to S and S' respectively. Then $\mathcal{C}_{ij} \xi_j' dS' = q_{ij} \xi_j dS$ where dS' and dS are the elements of area of S' and S respectively.

3. Strain Energy Function

In order to apply the theory mentioned we specify the strain energy function to be used. We use the usual strain energy function of the linear theory, namely $U = (\lambda/2)[e]^2 + \mu[e^2]$ where λ and μ are the lame constants, e is the strain matrix, and $[a]$ is the trace of the matrix a . Since we want U to be a function of p , we next define e as a function of p using a definition due to K. O. Friedrichs. Let c be the unique orthogonal matrix for which cp is symmetric and positive definite, and let $e = cp - 1$. We think of the rotation c^* as the rotational part of the deformation.

Next we wish to compute the matrix q . We observe that $(e + 1)^2 = p^*p$, $e^2 = p^*p - 2e - 1$, $[e^2] = [p^*p] - 2[e] - 3$, and $U = (\lambda/2)[e]^2 - 2\mu[e] + \mu[p^*p] - 3\mu$. Hence $q = (\lambda[e] - 2\mu)(\partial[e]/\partial p_{ik}) + 2\mu p$.

Next we show that $(\partial[e]/\partial p_{ik}) = c^*$. (1) From $cc^* = 1$ and $cp = p^*c^*$ we obtain $\partial c^*/\partial p_{ik} = -c^*(\partial c/\partial p_{ik})c^*$, and hence

$$\begin{aligned} \left[\frac{\partial c}{\partial p_{ik}} p \right] &= [p^* \frac{\partial c^*}{\partial p_{ik}}] = - [p^* c^* \frac{\partial c}{\partial p_{ik}} c^*] = - [cp \frac{\partial c}{\partial p_{ik}} c^*] = - \left[\frac{\partial c}{\partial p_{ik}} c^* cp \right] \\ &= - \left[\frac{\partial c}{\partial p_{ik}} p \right] \end{aligned}$$

so that $[\partial c/\partial p_{ik} p] = 0$. Thus $\partial[e]/\partial p_{ik} = \partial[cp]/\partial p_{ik} = [\partial c/\partial p_{ik} p] + [c(\partial p/\partial p_{ik})] = [c(\partial p/\partial p_{ik})] = c_{ki}$. We now have $q = (\lambda[e] - 2\mu)c^* + 2\mu p$.

We will also be working with linear combinations of derivatives with respect to the p_{ik} 's. For this we introduce a notation due to Fritz John [1]. For an arbitrary 3×3 real scalar matrix ξ let $D_\xi() = (\partial()/\partial p_{ik})\xi_{ik}$ where $()$ is either a scalar function or matrix function of the p_{ik} 's.

We now compute $D_\xi q_{ik}$ for arbitrary ξ evaluated for p equal a diagonal matrix with positive diagonal elements. For any p we have $D_\xi[e] = (\partial[e]/\partial p_{ij})\xi_{ij} = c_{ji}\xi_{ij} = [c\xi]$. When p is diagonal with positive diagonal elements, then $c = 1$ and hence

$$\begin{aligned} D_\xi q_{ik} &= (\lambda[e] - 2\mu)D_\xi c_{ki} + \lambda[c\xi]c_{ki} + 2\mu\xi_{ik} = (\lambda p_{jj} - 3\lambda - 2\mu)D_\xi c_{ki} \\ &\quad + \lambda\delta_{ik}\xi_{jj} + 2\mu\xi_{ik} \end{aligned}$$

From $cc^* = 1$ and $cp = p^*c^*$ we obtain $D_\xi c + D_\xi c^* = 0$ and $(D_\xi c)p + \xi = \xi^* + p(D_\xi c^*)$ for the p under consideration. Hence for such p we have $(D_\xi c)p + p(D_\xi c) = \xi^* - \xi$, and $D_\xi c_{ik} = \xi_{ki} - \xi_{ik}/p_{ii} + p_{kk}$.

We now have

$$D_{\xi} q_{ik} = \lambda \delta_{ik} \xi_{jj} + 2\mu \xi_{ik} + (\lambda p_{jj} - 3\lambda - 2\mu) \frac{\xi_{ik} - \xi_{ki}}{p_{ii} + p_{kk}},$$

for p diagonal with positive diagonal elements.

Below we list and number the results obtained thus far.

$$(3.1) \quad p = (p_{ij}) = (\partial x_i^1 / \partial x_j)$$

$$(3.2) \quad e = cp - 1, \quad c \text{ is orthogonal, } cp \text{ is symmetric and positive definite}$$

$$(3.3) \quad U = (\lambda/2)[e]^2 + \mu[e^2]$$

$$(3.4) \quad q = (\lambda[e] - 2\mu)c^* + 2\mu p$$

$$(3.5) \quad q_{ij,j} = 0 \quad (i = 1, 2, 3) \text{ for equilibrium with zero body forces}$$

$$(3.6) \quad \zeta = (1/|p|)qp^*$$

$$(3.7) \quad \zeta_{ij} \xi_j^1 dS^1 = q_{ij} \xi_j^1 dS \quad (i = 1, 2, 3)$$

$$(3.8) \quad D_{\xi} q_{ik} = \lambda \delta_{ik} \xi_{jj} + 2\mu \xi_{ik} + (\lambda p_{jj} - 3\lambda - 2\mu) \frac{\xi_{ik} - \xi_{ki}}{p_{ii} + p_{kk}}$$

for p diagonal with positive diagonal elements.

4. Statement of the Problem and Trivial Solution

We are now in a position to give a precise formulation of our problem. As before we let $\xi^1 = (\xi_1^1, \xi_2^1, \xi_3^1)$ be the unit outward normal vector to a material surface in the strained state, and we let $\xi = (\xi_1, \xi_2, \xi_3)$ be the unit outward normal vector to the same material surface in the unstrained state. In Lagrange coordinates we wish to solve the equilibrium equations $q_{ij,j} = 0 (i = 1, 2, 3)$ subject to the following boundary conditions:

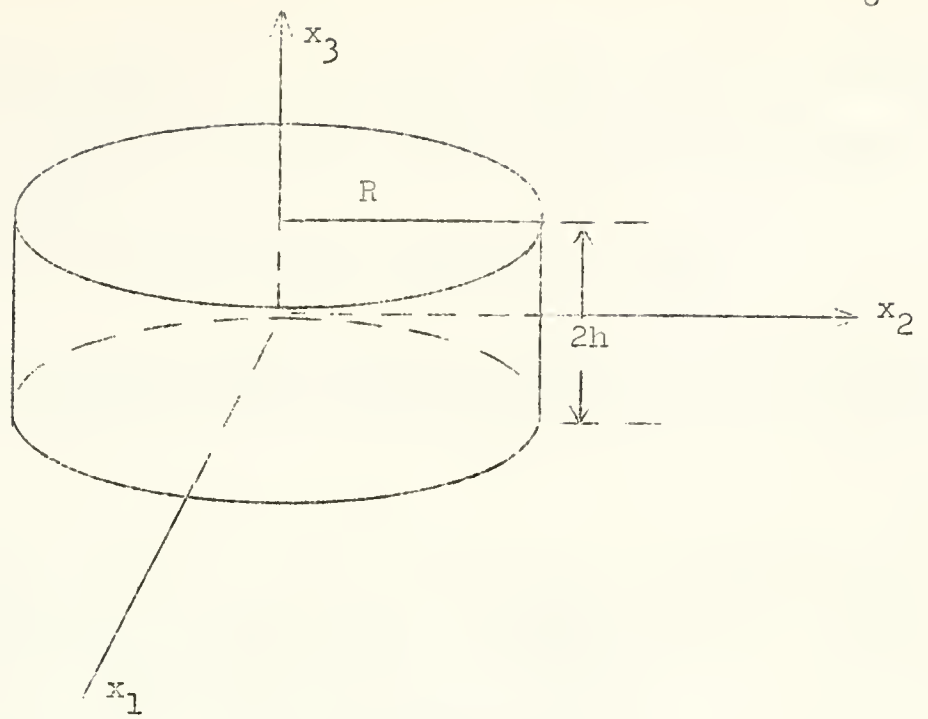


FIGURE I: Unrestrained State

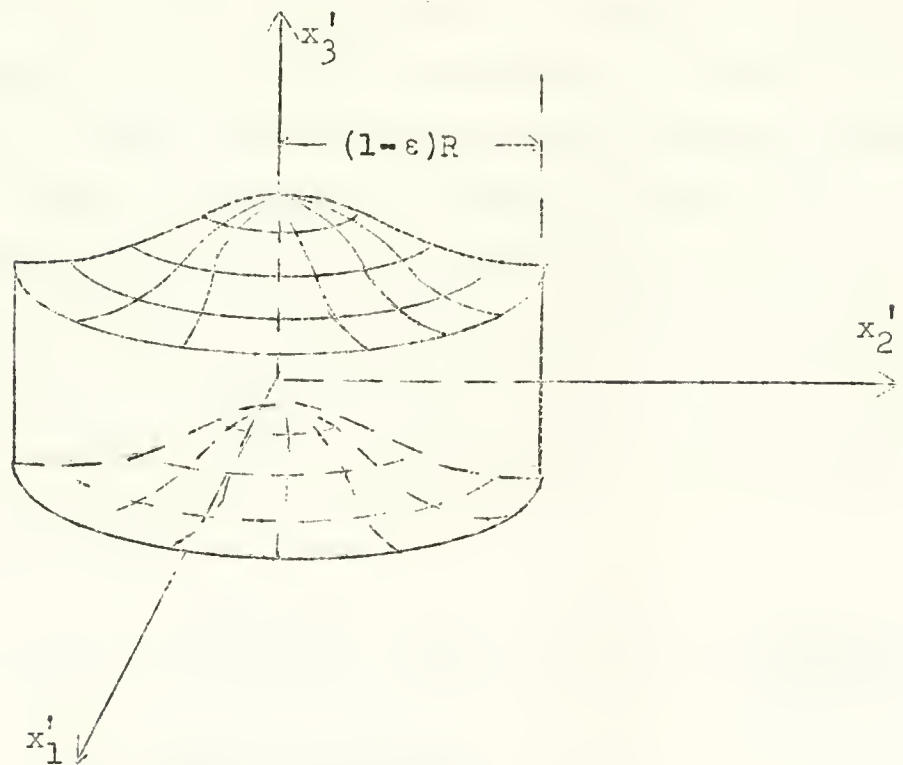


FIGURE II: First Buckled Mode

$$(4.1) \quad x_1' = (1 - \varepsilon)x_1, \quad x_2' = (1 - \varepsilon)x_2, \quad \text{and} \quad \tau_{3j}\xi_j' = 0 \quad \text{on the surface} \\ x_1 = R \cos \Theta \quad \text{and} \quad x_2 = R \sin \Theta \quad \text{where } \varepsilon \text{ is an arbitrary con-} \\ \text{stant such that } 0 < \varepsilon < 1.$$

$$(4.2) \quad \tau_{ij}\xi_j' = 0 \quad (i = 1, 2, 3) \quad \text{on the surfaces } x_3 = \pm h_3.$$

Using (3.7) we see that the boundary conditions can be expressed as follows:

$$(4.3) \quad x_1' = (1 - \varepsilon)x_1, \quad x_2' = (1 - \varepsilon)x_2, \quad \text{and} \quad q_{3j}\xi_j = q_{31}\cos \Theta + q_{32} \cdot \\ \sin \Theta = 0 \quad \text{for } x_1 = R \cos \Theta \quad \text{and} \quad x_2 = R \sin \Theta.$$

$$(4.4) \quad q_{ij}\xi_j = \pm q_{i3} = 0 \quad (i = 1, 2, 3) \quad \text{for } x_3 = \pm h.$$

First we look for a trivial solution of the form $x_i' = \delta_i x_i$ ($i = 1, 2, 3$) where each δ_i is a positive constant. For such a solution p is a constant diagonal matrix with positive diagonal elements. Hence $c = 1$ and by (3.4) q is a constant diagonal matrix. Hence the equilibrium equations are satisfied, $q_{31} \cos \Theta + q_{32} \sin \Theta = 0$ for $x_1 = R \cos \Theta$ and $x_2 = R \sin \Theta$, and $q_{13} = q_{23} = 0$ for $x_3 = \pm h$. From the other boundary conditions we obtain $\delta_1 = \delta_2 = 1 - \varepsilon$ and $0 = q_{33} = \lambda(\delta_1 + \delta_2 + \delta_3) - 3\lambda - 2\mu + 2\mu\delta_3$. Therefore $\delta_3 = 1 + (2\lambda\varepsilon/(\lambda+2\mu))$, and our trivial solution is

$$x_1' = (1 - \varepsilon)x_1, \quad x_2' = (1 - \varepsilon)x_2, \quad x_3' = (1 + \frac{2\lambda\varepsilon}{\lambda+2\mu})x_3.$$

5. Buckled Solutions

We now assume the existence of a solution in which x_i' ($i = 1, 2, 3$) and ε depend analytically on a parameter δ in the following way:

$$x_i' = \overset{\circ}{x}_i' + \overset{\bullet}{x}_i'\delta + \dots \quad \text{and} \quad \varepsilon = \varepsilon_0 + \varepsilon\delta + \dots$$

where $\overset{\circ}{x}_i^!$ ($i = 1, 2, 3$) is the trivial solution corresponding to $\varepsilon = \varepsilon_0$ and where $x_i^!$ ($i = 1, 2, 3$) is non-trivial for $\delta > 0$. Then $x_i^!$ ($i = 1, 2, 3$) is a trivial solution for $\delta = 0$ and a buckled solution for $\delta > 0$.

Our objective now is to find the possible values of ε_0 (i.e., the values of ε at which buckling can start).

Let $(\overset{\circ}{})$ be the value of () at $\delta = 0$, and let $(\overset{\circ}{})$ be the value of $d()/d\delta$ at $\delta = 0$.

Let $a = 1 - \varepsilon_0$ and $b = 1 + (2\lambda\varepsilon_0/\lambda + 2\mu)$ so that $x_1^! = ax_1$, $x_2^! = ax_2$, and $x_3^! = bx_3$.

Taking the derivative with respect to δ at 0 in the equilibrium equations and boundary conditions (4.3) - (4.4) we obtain

$$(5.1) \quad \overset{\circ}{q}_{ik,k} = 0 \quad (i = 1, 2, 3)$$

$$(5.2) \quad \overset{\circ}{x}_1^! = -\overset{\circ}{\varepsilon}x_1, \quad \overset{\circ}{x}_2^! = -\overset{\circ}{\varepsilon}x_2, \quad \text{and} \quad \overset{\circ}{q}_{31} \cos \theta + \overset{\circ}{q}_{32} \sin \theta = 0 \quad \text{for} \\ x_1 = R \cos \theta \quad \text{and} \quad x_2 = R \sin \theta.$$

$$(5.3) \quad \overset{\circ}{q}_{i3} = 0 \quad (i = 1, 2, 3) \quad \text{for} \quad x_3 = \pm h.$$

Next we observe that

$$\overset{\circ}{q}_{ik,k} = \frac{\partial \overset{\circ}{q}_{ik}}{\partial x_k} = \frac{\partial \overset{\circ}{q}_{ik}}{\partial p_{rs}} p_{rs,k},$$

and hence

$$\begin{aligned} \overset{\circ}{q}_{ik,k} &= \frac{d}{d\delta} \left(\frac{\partial \overset{\circ}{q}_{ik}}{\partial p_{rs}} \right) \Big|_{\delta=0} \overset{\circ}{p}_{rs,k} + \left(\frac{\partial \overset{\circ}{q}_{ik}}{\partial p_{rs}} \right)_{\delta=0} \overset{\circ}{p}_{rs,k} = \left(\frac{\partial \overset{\circ}{q}_{ik}}{\partial p_{rs}} \right)_{\delta=0} \overset{\circ}{p}_{rs,k} \\ &= (D_{\xi_k} \overset{\circ}{q}_{ik})_{\delta=0} \quad \text{for} \quad \xi_k = \overset{\circ}{p}_{,k}. \end{aligned}$$

Also $\dot{q}_{ik} = (\partial q_{ik} / \partial p_{rs})_{\delta=0} \dot{p}_{rs} = (D_{\xi} q_{ik})_{\delta=0}$ where $\xi = \dot{p}$.

We now modify (5.1) - (5.3) using (3.8) and the fact that $\dot{p}$ is a diagonal matrix with positive diagonal elements. Observing that $\lambda \dot{p}_{jj} - 3\lambda - 2\mu = -2\mu b$ we obtain from (5.1)

$$0 = (D_{\xi_k} q_{ik})_{\delta=0} = \lambda \delta_{ik} \dot{p}_{jj,k} + 2\mu \dot{p}_{ik,k} - \frac{2\mu b}{\dot{p}_{ii} + \dot{p}_{kk}} (\dot{p}_{ik,k} - \dot{p}_{ki,k}) .$$

From (5.2) when $x_1 = R \cos \theta$ and $x_2 = R \sin \theta$ we have

$$0 = (D_{\xi} q_{31})_{\delta=0} \cos \theta + (D_{\xi} q_{32})_{\delta=0} \sin \theta = \frac{2\mu}{a+b} (a \dot{p}_{31} + b \dot{p}_{13}) \\ \cdot \cos \theta + \frac{2\mu}{a+b} (a \dot{p}_{32} + b \dot{p}_{23}) \sin \theta ,$$

and $\dot{p}_{31} \cos \theta + \dot{p}_{32} \sin \theta = 0$ since here

$$\dot{p}_{13} = (-\dot{\epsilon} x_1)_{x_3} = 0 \quad \text{and} \quad \dot{p}_{23} = (-\dot{\epsilon} x_2)_{x_3} = 0 .$$

From (5.3) when $x_3 = \pm h$ we have

$$0 = (D_{\xi} q_{13})_{\delta=0} = \frac{2\mu}{a+b} (a \dot{p}_{13} + b \dot{p}_{31}) ,$$

$$0 = (D_{\xi} q_{23})_{\delta=0} = \frac{2\mu}{a+b} (a \dot{p}_{23} + b \dot{p}_{32}) ,$$

and

$$0 = (D_{\xi} q_{33})_{\delta=0} = \lambda (\dot{p}_{11} + \dot{p}_{22}) + (\lambda + 2\mu) \dot{p}_{33} .$$

Listing these results we have

$$(5.4) \quad \lambda \dot{p}_{jj,i} + 2\mu \dot{p}_{ik,k} - \frac{2\mu b}{\dot{p}_{ii} + \dot{p}_{kk}} (\dot{p}_{ik,k} - \dot{p}_{ki,k}) = 0 \quad (i = 1, 2, 3).$$

$$(5.5) \quad \dot{x}'_1 = -\dot{e}x_1, \quad \dot{x}'_2 = -\dot{e}x_2, \quad \dot{p}_{31} \cos \theta + \dot{p}_{32} \sin \theta = 0 \text{ for } x_1 = R \cos \theta \text{ and } x_2 = R \sin \theta.$$

$$(5.6) \quad 0 = a\dot{p}_{13} + b\dot{p}_{31} = a\dot{p}_{23} + b\dot{p}_{32} = \lambda(\dot{p}_{11} + \dot{p}_{22}) + (\lambda + 2\mu)\dot{p}_{33} \text{ for } x_3 = \pm h.$$

To simplify the problem we assume at this point that our solution has radial symmetry. That is, we assume that $x'_1 = f(r, x_3) \cos \theta$, $x'_2 = f(r, x_3) \sin \theta$, and $x'_3 = g(r, x_3)$ for $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$ and for some functions f and g .

Then $\dot{f} = ar$, $\dot{g} = bx_3$, $\dot{x}'_1 = \dot{f} \cos \theta$, $\dot{x}'_2 = \dot{f} \sin \theta$, and $\dot{x}'_3 = \dot{g}$.

Also

$$\dot{p} = \begin{pmatrix} \dot{f}_r \cos^2 \theta + \frac{1}{r} \dot{f} \sin^2 \theta & (\dot{f}_r - \frac{1}{r} \dot{f}) \sin \theta \cos \theta & \dot{f}_{x_3} \cos \theta \\ (\dot{f}_r - \frac{1}{r} \dot{f}) \sin \theta \cos \theta & \dot{f}_r \sin^2 \theta + \frac{1}{r} \dot{f} \cos^2 \theta & \dot{f}_{x_3} \sin \theta \\ \dot{g}_r \cos \theta & \dot{g}_r \sin \theta & \dot{g}_{x_3} \end{pmatrix}$$

Let $A = 2\mu a/a+b$ and $B = \lambda + (2\mu b/a+b)$.

Using the fact that now $\dot{p}_{21} = \dot{p}_{12}$ we obtain (5.7) from (5.4).

$$(5.7) \quad \begin{cases} (\lambda + 2\mu)(\dot{p}_{11,1} + \dot{p}_{12,2}) + A\dot{p}_{13,3} + B\dot{p}_{31,3} = 0 \\ (\lambda + 2\mu)(\dot{p}_{21,1} + \dot{p}_{22,2}) + A\dot{p}_{23,3} + B\dot{p}_{32,3} = 0 \\ A(\dot{p}_{31,1} + \dot{p}_{32,2}) + (\lambda + 2\mu)\dot{p}_{33,3} + B(\dot{p}_{11,3} + \dot{p}_{22,3}) = 0 \end{cases}$$

Making full use of radial symmetry in (5.7), (5.5), and (5.6) we obtain

$$(5.8) \quad \begin{cases} (\lambda + 2\mu)(\dot{f}_{rr} + \frac{1}{r}\dot{f}_r - \frac{1}{r^2}\dot{f}) + A\dot{f}_{x_3x_3} + B\dot{g}_{rx_3} = 0 \\ A(\dot{g}_{rr} + \frac{1}{r}\dot{g}_r) + (\lambda + 2\mu)\dot{g}_{x_3x_3} + B(\dot{f}_{rx_3} + \frac{1}{r}\dot{f}_{x_3}) = 0 \end{cases}$$

$$(5.9) \quad \dot{f}(R, x_3) = -\dot{\epsilon}R, \quad \dot{g}_r(R, x_3) = 0$$

$$(5.10) \quad a\dot{f}_{x_3} + b\dot{g}_r = 0, \quad \lambda(\dot{f}_r + \frac{1}{r}\dot{f}) + (\lambda + 2\mu)\dot{g}_{x_3} = 0 \quad \text{for } x_3 = \pm h.$$

We look for a series solution of (5.8) having the form $\dot{f} = \sum_{n=0}^{\infty} a_n(x_3)\gamma_n(r)$ and $\dot{g} = \sum_{n=0}^{\infty} \beta_n(x_3)\delta_n(r)$. When we substitute these series into (5.8) we would like to be able to express the contribution made by the n th terms as a product of a function of x_3 and a function of r . For this to be possible it is sufficient that $\gamma_n'' + (1/r)\gamma_n' - (1/r^2)\gamma_n$, γ_n , and δ_n' are proportional and also that $\delta_n'' + (1/r)\delta_n'$, δ_n , and $\gamma_n' + (1/r)\gamma_n$ are proportional. These proportionality conditions are satisfied if $\gamma_n = J_1(k_n r)$ and $\delta_n = J_0(k_n r)$ where k_n is an arbitrary constant and J_n is the Bessel function of the first kind of order n . Since a wide class of functions can be expanded in series of the functions $J_1(k_n r)$ or $J_0(k_n r)$, and since we can easily satisfy (5.9) term by term for such series, we look for solutions of the form

$$\dot{f} = -\dot{\epsilon}r + \sum_{n=1}^{\infty} a_n(x_3)J_1(k_n r)$$

and

$$\dot{g} = \beta_0(x_3) + \sum_{n=1}^{\infty} \beta_n(x_3)J_0(k_n r)$$

where k_n is chosen so that $k_n R$ is the n th positive zero of J_1 .

Using the identity $J_0' = -J_1$ and the differential equation for J_0 we obtain $(d/dr)J_0(k_n r) = -k_n J_1(k_n r)$ and

$(d/dr)J_1(k_n r) = -k_n J_0'(k_n r) = (1/r)J_0'(k_n r) + k_n J_0(k_n r) = - (1/r)J_1(k_n r) + k_n J_0(k_n r)$. With these results we obtain

$$\dot{f}_r = -\dot{\varepsilon} + \sum_{n=1}^{\infty} a_n \left[-\frac{1}{r} J_1(k_n r) + k_n J_0(k_n r) \right] ,$$

$$\dot{f}_{rr} = \sum_{n=1}^{\infty} a_n \left[\frac{2}{r^2} J_1(k_n r) - k_n^2 J_1(k_n r) - \frac{k_n}{r} J_0(k_n r) \right] ,$$

$$\dot{g}_r = - \sum_{n=1}^{\infty} k_n \beta_n J_1(k_n r) ,$$

and

$$\dot{g}_{rr} = \sum_{n=1}^{\infty} k_n \beta_n \left[\frac{1}{r} J_1(k_n r) - k_n J_0(k_n r) \right] .$$

Thus

$$\dot{f}_r + \frac{1}{r} \dot{f} = -2\dot{\varepsilon} + \sum_{n=1}^{\infty} k_n a_n J_0(k_n r) ,$$

$$\dot{f}_{rr} + \frac{1}{r} \dot{f}_r - \frac{1}{r^2} \dot{f} = - \sum_{n=1}^{\infty} k_n^2 a_n J_1(k_n r) ,$$

and

$$\dot{g}_{rr} + \frac{1}{r} \dot{g}_r = - \sum_{n=1}^{\infty} k_n^2 \beta_n J_0(k_n r) .$$

Using these results in (5.8) - (5.10) we obtain:

$$(5.11) \quad \begin{cases} -(\lambda + 2\mu)k_n^2 a_n + Aa_n'' - Bk_n \beta_n' = 0 & \text{for } n = 1, 2, \dots \\ -Ak_n^2 \beta_n + (\lambda + 2\mu)\beta_n'' + Bk_n a_n' = 0 & \text{for } n = 1, 2, \dots \\ \beta_0'' = 0 \end{cases}$$

$$(5.12) \quad \begin{cases} a\alpha_n' - bk_n\beta_n = 0 \quad (n = 1, 2, \dots) & \text{for } x_3 = \pm h \\ \lambda k_n \alpha_n + (\lambda + 2\mu)\beta_n' = 0 \quad (n = 1, 2, \dots) & \text{for } x_3 = \pm h \\ -2\lambda\varepsilon + (\lambda + 2\mu)\beta_0' = 0 & \text{for } x_3 = \pm h \end{cases}$$

From this we easily obtain $\beta_0 = (2\lambda\varepsilon/\lambda+2\mu) + \beta_{00}$ for some constant β_{00} . Clearly the value of β_{00} is determined by the displacement of the whole plate parallel to the x_3 -axis, and without loss of generality we may choose $\beta_{00} = 0$.

Eliminating β_n from the first two equations of (5.11) we obtain $\alpha_n^{(4)} - 2k_n^2\alpha_n'' + k_n^4\alpha_n = 0$ so that

$$\alpha_n = (c_1x_3 + c_2) \sinh k_nx_3 + (c_3x_3 + c_4) \cosh k_nx_3.$$

It then follows easily that

$$\beta_n = [- (c_3x_3 + c_4) + \frac{2A+B}{Bk_n} c_1] \sinh k_nx_3 + [- (c_1x_3 + c_2) + \frac{2A+B}{Bk_n} c_3] \cdot \cosh k_nx_3.$$

The first two equations of (5.12) now become

$$(5.13) \quad \begin{cases} [(a+b)k_n(c_3x_3 + c_4) + \frac{aB-b(2A+B)}{B} c_1] \sinh k_nx_3 \\ + [(a+b)k_n(c_1x_3 + c_2) + \frac{aB-b(2A+B)}{B} c_3] \cosh k_nx_3 = 0 \\ \text{for } x_3 = \pm h_3 \\ [-2\mu k_n(c_1x_3 + c_2) + \frac{2A(A+B)}{B} c_3] \sinh k_nx_3 \\ + [-2\mu k_n(c_3x_3 + c_4) + \frac{2A(A+B)}{B} c_1] \cosh k_nx_3 = 0 \text{ for } x_3 = \pm h_3 \end{cases}$$

We note that in (5.13) we have four linear homogeneous equations in the unknowns c_1, c_2, c_3, c_4 . We make the assumption that $\alpha_n \neq 0$ and $\beta_n \neq 0$ for some $n \geq 1$ so that our solution differs from the trivial solution to the first order in δ . Then for some n we have c_1, c_2, c_3 , and c_4 are not all zero, and from (5.13) we obtain a determinant which must be zero. Simplifying that determinant equation we obtain

$$\frac{\sinh 2k_n h}{2k_n h} = \pm \frac{\lambda(2+\epsilon_0)+2\mu}{\lambda(2-5\epsilon_0)+2\mu(1-2\epsilon_0)}.$$

Since we want our result to make sense for plates when ϵ_0 is small, we choose the plus sign. Solving for ϵ_0 we have

$$(5.14) \quad \epsilon_0 = \frac{2(\lambda+\mu) \left[\frac{\sinh 2k_n h}{2k_n h} - 1 \right]}{\lambda + (5\lambda+4\mu) \frac{\sinh 2k_n h}{2k_n h}}.$$

Let σ be the Poisson ratio, let D be the depth or thickness of the plate, and let j_n be the n th positive zero of J_1 . Then $\sigma = \lambda/2(\lambda+\mu)$, $D = 2h$, and $k_n = j_n/R$. We can now write (5.14) as

$$(5.15) \quad \epsilon_0 = \frac{\frac{R}{j_n D} \sinh \frac{j_n D}{R} - 1}{\sigma + (2+\sigma) \frac{R}{j_n D} \sinh \frac{j_n D}{R}}.$$

Since (5.14) can be valid for at most one value of n for each value of ϵ_0 , our series representations of $\dot{f}$ and $\dot{g}$ become the finite series

$$(5.16) \quad \dot{f} = -\dot{\epsilon} r + \alpha_n J_1(k_n r), \quad \dot{g} = \beta_0 + \beta_n J_0(k_n r)$$

where n may be any positive integer, and ϵ_0 is related to n by (5.14).

Using (5.14) and (5.13) and solving for c_1, c_2, c_3 , and c_4 we obtain $c_1 = 0$, $c_2 = -w\bar{c}$, $c_3 = \bar{c}$, and $c_4 = 0$, where $\bar{c}$ is a constant (not determined by (5.13)), and $w = h \tanh k_n h + (aB - b(2A+B))/Bk_n(a+b)$.

Listing our results for the n th mode of buckling we have

$$(5.17) \quad \begin{cases} \alpha_n = \bar{c}(-\sigma \sinh k_n x_3 + x_3 \cosh k_n x_3) \\ \beta_n = \bar{c}[-x_3 \sinh k_n x_3 + (\frac{2A+B}{Bk_n} + w) \cosh k_n x_3] \\ x'_1 = ax_1 + [-\dot{\epsilon}x_2 + \alpha_n \frac{x_1}{r} J_1(k_n r)]\delta + O(\delta^2) \\ x'_2 = ax_2 + [-\dot{\epsilon}x_2 + \alpha_n \frac{x_2}{r} J_1(k_n r)]\delta + O(\delta^2) \\ x'_3 = bx_3 + [\frac{2\lambda\dot{\epsilon}x_3}{\lambda+2\mu} + \beta_n J_0(k_n r)]\delta + O(\delta^2) \end{cases}$$

Now let P_n be the n -th critical buckling pressure. Clearly

$$P_n = -\frac{\overset{\circ}{T}}{\overset{\circ}{p}}_{11} \text{ so that from (3.6) we have } P_n = -\frac{1}{|\overset{\circ}{p}|} \overset{\circ}{q}_{ij} \overset{\circ}{p}_{1j} = -\frac{1}{a^2 b} \overset{\circ}{q}_{11} \overset{\circ}{p}_{11} = -\frac{1}{ab} \overset{\circ}{q}_{11}. \text{ From (3.4) we have } \overset{\circ}{q}_{11} = -2\mu b + 2\mu a \text{ so that}$$

$$P_n = 2\mu \frac{b-a}{ab} = \frac{2\mu(3\lambda+2\mu)\epsilon_0}{\lambda+2\mu+(\lambda-2\mu-2\lambda\epsilon_0)\epsilon_0} = \frac{E\epsilon_0}{1-\sigma+(3\sigma-1)\epsilon_0-2\sigma\epsilon_0^2}$$

where E is Young's modulus of elasticity and σ is the Poisson ratio.

We list this as

$$(5.18) \quad P_n = \frac{\epsilon_0 E}{1-\sigma+(3\sigma-1)\epsilon_0-2\sigma\epsilon_0^2}$$

where ϵ_0 is given by (5.14) and (5.15).

It is interesting to notice that ϵ_0 and P_n both approach finite limits (not dependent on n) as $\frac{D}{R} \rightarrow \infty$. Hence for a certain finite compression of the cylinder we expect all modes of buckling to be

solutions of the mathematical model regardless of the thickness of the plate.

6. Comparison with Linear Theory Results

For thin plates ϵ_0 is arbitrarily small and from (5.18) we have

$$P_n = \frac{\epsilon_0 E}{1-\sigma} \sum_{k=0}^{\infty} \left(\frac{1-3\sigma+2\sigma^2\epsilon_0}{1-\sigma} \right)^k \epsilon_0^k .$$

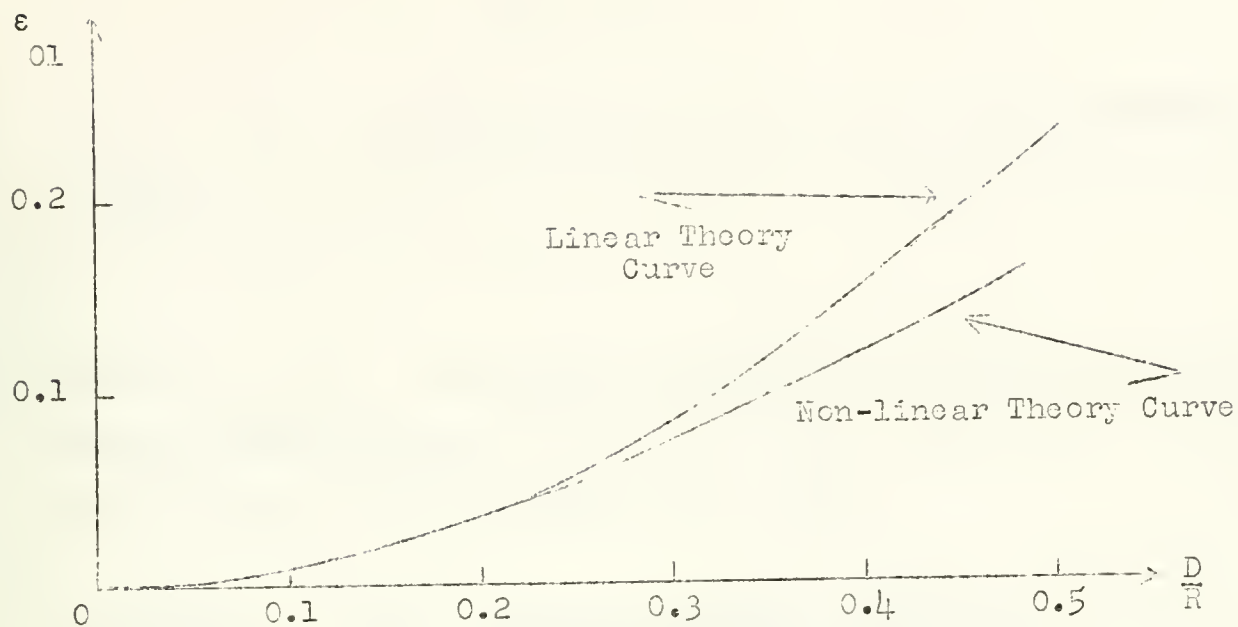
Retaining second order terms in D/R in (5.15) we obtain $\epsilon_0 = (1/12(1+\sigma))(j_n D/R)^2$. Thus retaining second order terms in D/R in the formula for P_n we have $P_n = (E/12(1-\sigma^2))(j_n D/R)^2$. If we let F be the flexural rigidity of the plate, then $F = ED^3/12(1-\sigma^2)$ and $P_n = Fj_n^2/DR^2$. Hence the n th critical buckling pressure per unit circumference, $P_n D$, is given by

$$(6.1) \quad P_n D = \frac{Fj_n^2}{R^2}$$

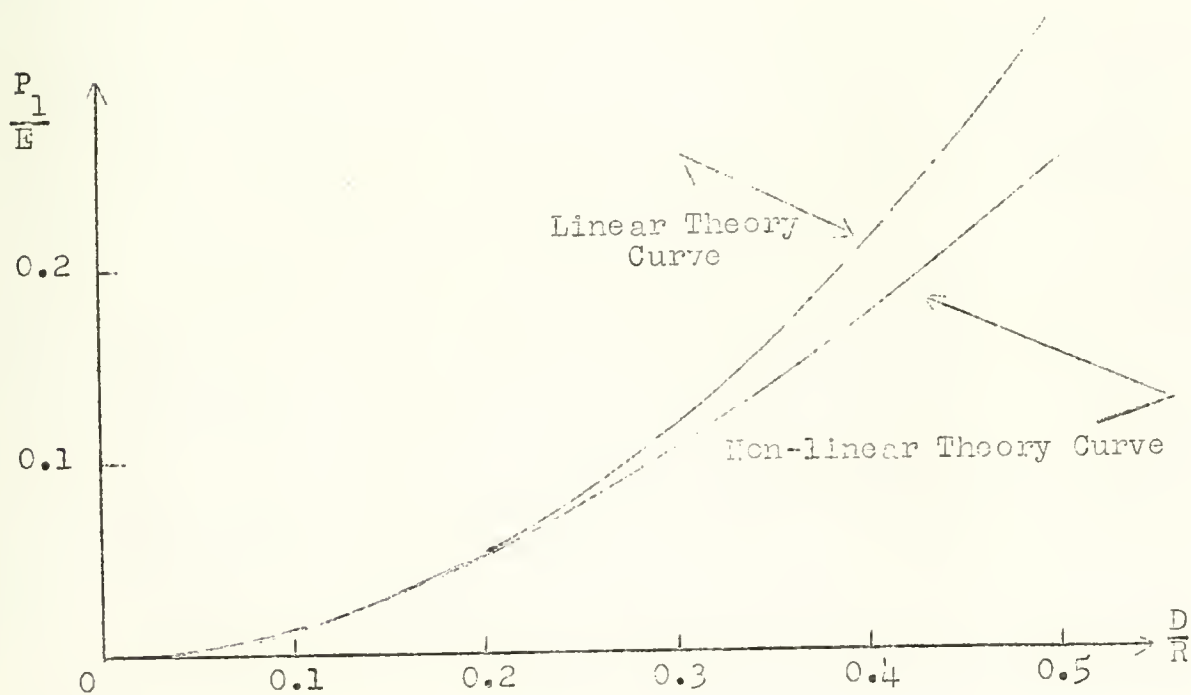
which is the result obtained using the linear theory for a plate with clamped edges [3].

In graphs I and II we make a graphical comparison between the results of the linear and nonlinear theories. Let ϵ_{01} be the value of ϵ_0 for $n = 1$. Graphs I and II show how ϵ_{01} and P_1/E respectively vary with D/R for the linear and nonlinear theories when $\sigma = 0.3$.

If we consider the value of ϵ_{01} given by the nonlinear theory as the correct value, and if we think of the value given by the linear theory as an approximate value, then for $\sigma = 0.3$ the per cent error due to the linear theory is 1.4 per cent for $D/R = 0.1$ and 5.8 per cent for $D/R = 0.2$. Thus the per cent error is small for $D/R \leq 0.1$, and it approaches zero as $D/R \rightarrow 0$.



GRAPH I: Poisson ratio = 0.3



GRAPH II: Poisson ratio = 0.3

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